Singular perturbation problems arising in mathematical finance: fluid dynamics concepts in option pricing

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comprehensive, even epic, work *Paul Wilmott on Quantitative Finance*. It includes carefully selected chapters to give the student a thorough understanding of futures, options and numerical methods. New software has been added and "Time Out" sections included which explain the mathematics for those less confident in this area.

In praise of Paul Wilmott and his previous works

"This book provides a very readable introduction to the quantitative methods/models that underlie the modern-day world of derivative contract valuation and risk management. The only prerequisite is a working knowledge of basic calculus. Wilmott does the rest. The text is free-flowing, with plenty of figures, illustrations, and applications that help build the reader's economic intuition for the subject matter. The book is ideally suited for advanced undergraduate and graduate courses in economics and finance."
ROBERT E. WHALEY, T. AUSTIN FINCH FOUNDATION PROFESSOR, DUKE UNIVERSITY

"It is a serious work that takes the reader all the way from the simplest of notions to the most complicated of recent models. In short it is the most comprehensive and up to date textbook on options that I have seen ... The style is jocular, but the content heavyweight. ... Who ever heard of a mathematician who could convey the intuition of a result to those with a less complete training in the subject? Wilmott is an exception: he knows when a result is hard to understand and treats the reader in a sympathetic manner ... This book is a splendid achievement." THE TIMES HIGHER EDUCATIONAL SUPPLEMENT

"...a text which will probably come to rank alongside Fabozzi's collected works of Leibowitz as a comprehensive practical reference source for financial theory." FUTURES AND OTC WORLD

"Paul Wilmott has produced one of the most exciting and classic reference volumes on derivatives which is a must for ... students, practitioners, risk managers." GLOBAL TRADING

"As only great teachers can, Wilmott makes even the most obtuse mathematics seem easy and intuitive."
MARCO AVELLANEDA, PROFESSOR OF MATHEMATICS AND DIRECTOR, DIVISION OF QUANTITATIVE FINANCE, COURANT INSTITUTE OF MATHEMATICAL SCIENCE, NEW YORK UNIVERSITY

"Paul Wilmott changed my life." DAVID NEWTON, MANCHESTER BUSINESS SCHOOL

See inside for less reverent opinions.

Paul Wilmott, described by the *Financial Times* as "cult derivatives lecturer," is one of the world's leading experts on quantitative finance and derivatives.

A crash course in singular perturbation theory

Consider the ordinary differential equation:

$$\epsilon \frac{d^2 y}{dx^2} - y = 1$$

subject to $y(0) = 0$, $y(1) = -1$.
Standard (undergraduate) methods lead to the exact solution:

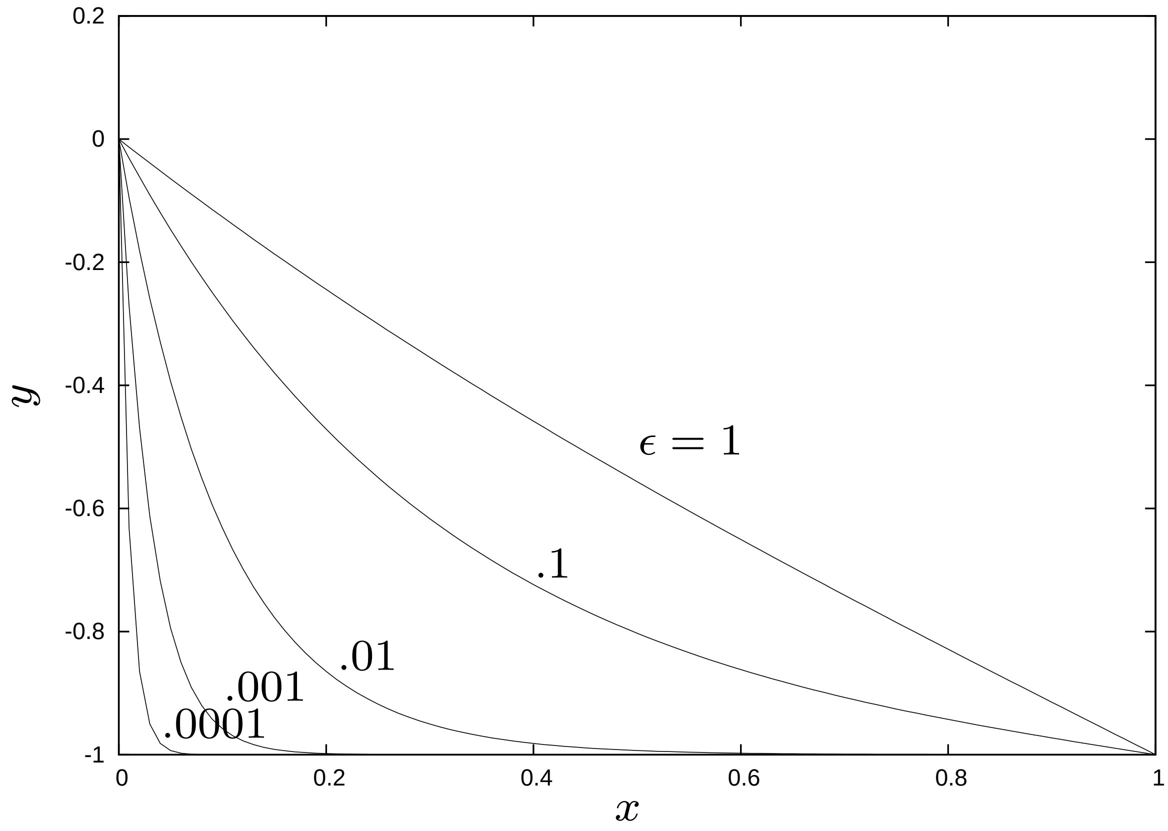
$$y = -1 + \frac{1}{1 - e^{2/\sqrt{\epsilon}}} \left(e^{2/\sqrt{\epsilon} - x/\sqrt{\epsilon}} - e^{x/\sqrt{\epsilon}} \right)$$

Consider another approach *if* ϵ *becomes small*.
Setting $\epsilon = 0$ in the ODE leads immediately to

$$y = -1$$

- This clearly satisfi es the boundary condition at $x = 1$
- This clearly violates the boundary condition at $x = 0$

Exact solution variation with ϵ



Boundary-layer concept

Exact results suggest, as $\epsilon \to 0$:

- $y \approx -1$ everywhere, except
- Close to $x = 0$, steep solution gradient

Suggests the concept of a boundary layer - the derivative term ($\frac{d^2y}{dx^2}$) term can no longer be neglected.

Suggests need $\frac{d^2}{dx^2} = O(\epsilon^{-1})$, i.e.

$$x = O(\sqrt{\epsilon})$$

Defi ne $X = x/\sqrt{\epsilon}$, then differential equation becomes

$$\frac{d^2y}{dX^2} - y = 1$$

subject to $y(X = 0) = 0$, $y(X \to \infty) \to -1$

The solution (continued)

Solution (marginally†) simpler than full problem, leads to

$$y = -1 + e^{-X}$$

i.e.

$$y = -1 + e^{-x/\sqrt{\epsilon}}$$

This agrees with exact solution as $\epsilon \to 0$
Singular perturbation problems typically arise from the neglect of the highest order derivative, which leads to ‘diffi culties’ in thin zones - boundary layers, shear layers but perfectly acceptable solutions outside these zones
If $y(1) \neq -1$, then another boundary layer exists, near $x = 1$.

† This is a Mickey Mouse example

THE LIMIT OF SMALL VOLATILITY

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Typical value of volatility $\sigma \approx 0.1 - 0.4$ (per $(\mathrm{annum})^{\frac{1}{2}}$)
- Typical value of interest rates $r \approx 0.01 - 0.1$ (per annum)
- BSE has σ^2 multiplying highest order (S) derivative - conditions ripe for SINGULAR PERTURBATION PROBLEM
- Consider a European (call) option

Simply setting $\sigma = 0$ leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} - rC = 0$$

Final condition: $C(S,T) = max(S - E, 0)$
Solution is

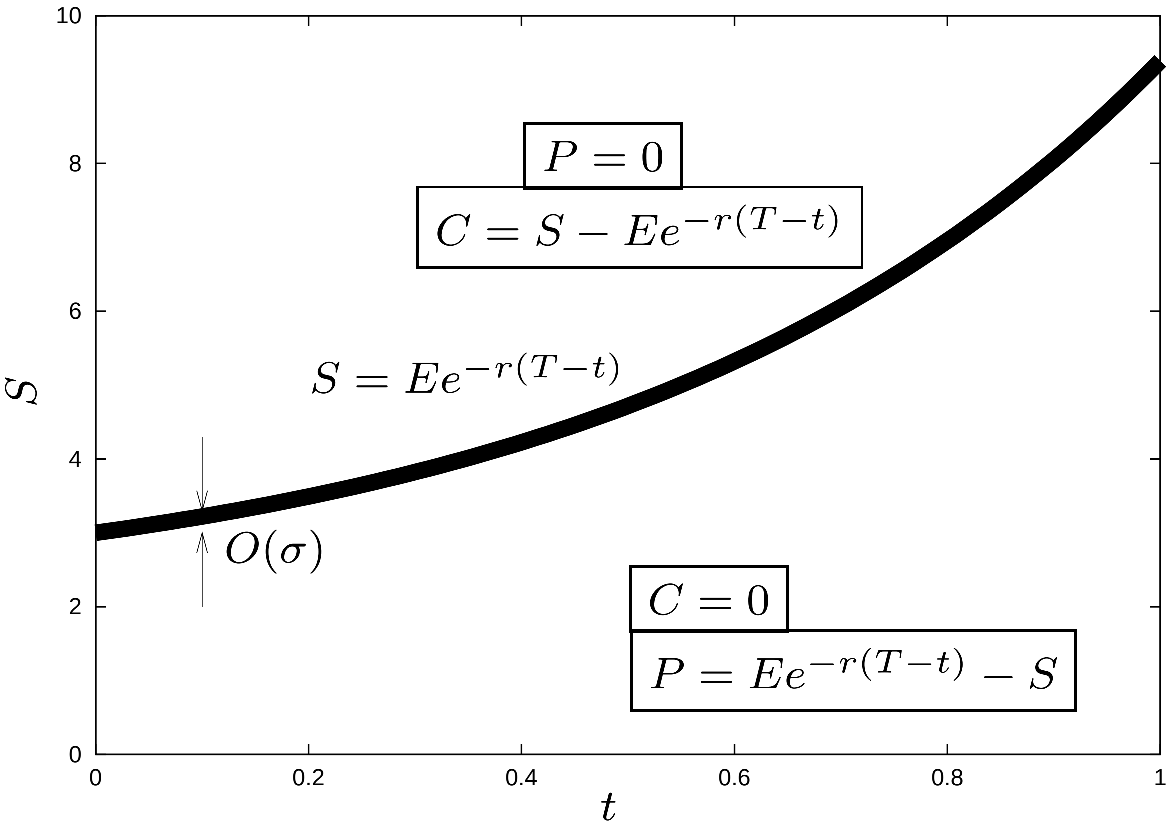
$$C \equiv 0 \quad \text{if } S - Ee^{-r(T-t)} < 0$$
$$C = S - Ee^{-r(T-t)} \quad \text{if } S - Ee^{-r(T-t)} > 0$$

• Discontinuity (in slope) at $S = Ee^{-r(T-t)}$ (at the forward money)

The shear layer

- Asymptotic balancing suggests *shear layer* † to smooth out discontinuity - along $S = Ee^{-r(T-t)}$ - *at the forward money*
- Thickness $O(\sigma)$, so that $\sigma^2 \frac{\partial^2}{\partial S^2} \sim \frac{\partial}{\partial t}$
- Defi ne $\hat{S} = (S - Ee^{-r(T-t)})/\sigma = O(1)$, $\hat{C} = C/\sigma = O(1)$
- $O(\sigma)$ equation: $\mathcal{L}\{\hat{C}\} \equiv r\hat{S}\frac{\partial \hat{C}}{\partial \hat{S}} + \frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\left[Ee^{-r(T-t)}\right]^2 \frac{\partial^2 \hat{C}}{\partial \hat{S}^2} - r\hat{C} = 0$
- $\hat{C} \to 0$ as $\hat{S} \to -\infty$, $\hat{C} \to \hat{S}$ as $\hat{S} \to \infty$.
- Corresponding put option: $\hat{P} \to 0$ as $\hat{S} \to \infty$, $\hat{P} \to -\hat{S}$ as $\hat{S} \to -\infty$.

† A thin region separating two mis-matching (outer) regions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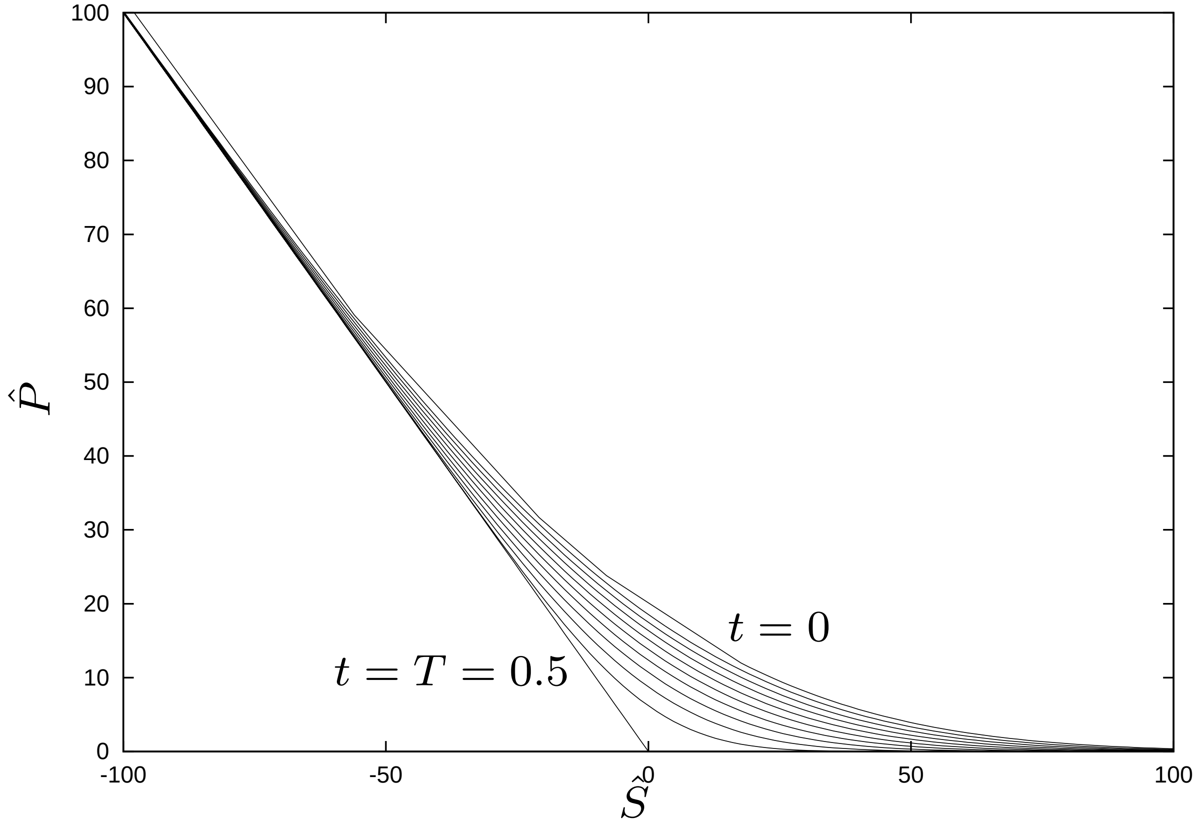
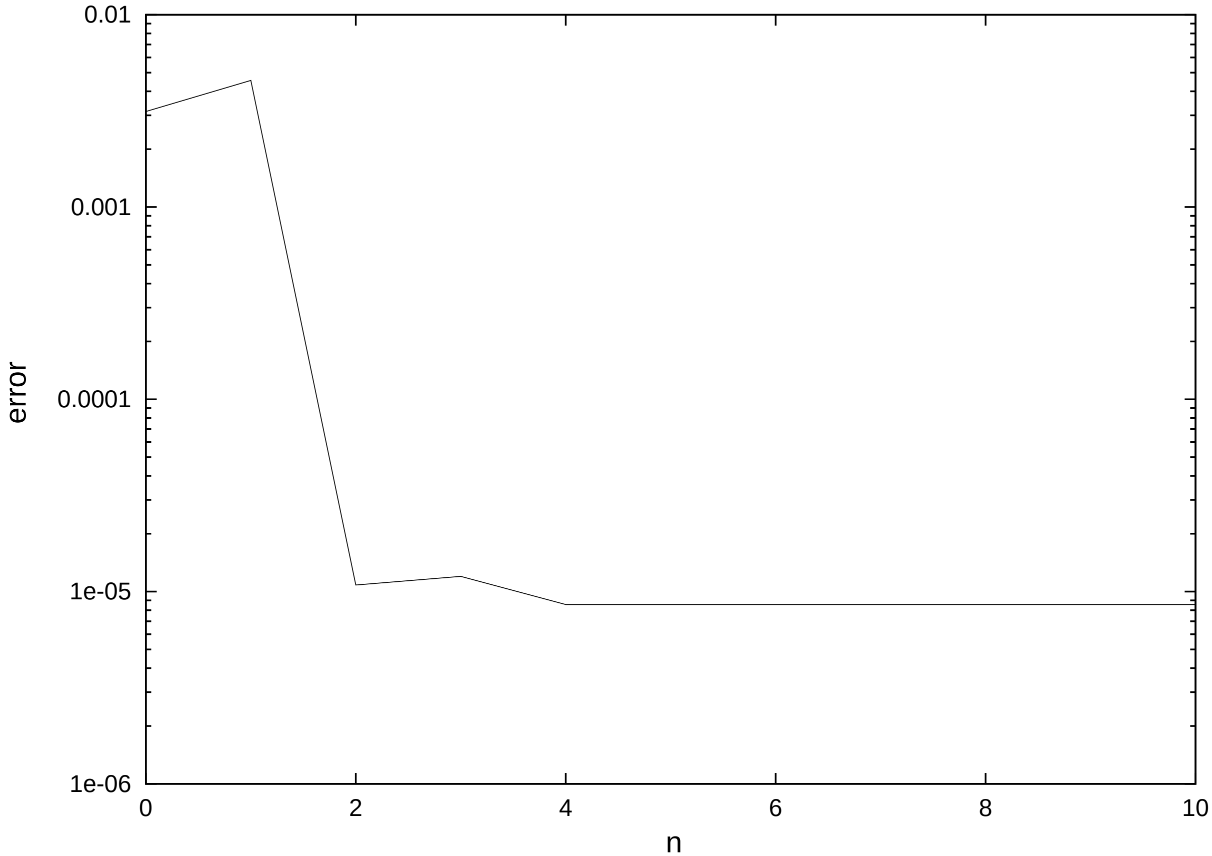
$P = 0$
$C = S - Ee^{-r(T-t)}$
$S = Ee^{-r(T-t)}$
$O(\sigma)$
$C = 0$
$P = Ee^{-r(T-t)} - S$



Put option, $E = 71$, $r = 0.05$, $T = 0.5$

Can continue asymptotic expansion *ad nauseam* by $V = \sum_{n=1}^{\infty} \sigma^n \hat{V}_n(\hat{S}, t)$
For $n \geq 2$:

$$\mathcal{L}\{\hat{V}_n\} = -\hat{S}Ee^{-r(T-t)}\frac{\partial^2 \hat{V}_{n-1}}{\partial \hat{S}^2} - \frac{1}{2}\hat{S}^2\frac{\partial^2 \hat{V}_{n-2}}{\partial \hat{S}^2}$$



Call option, $E = 100$, $r = 0.06$, $T = 0.5$, $S = E^{-rT}$, $\sigma = 0.2$

Barrier/down and out (put) options - CASE I

Option worthless if underlying hits/drops below prescribed value B
At expiry, $P(S,T) = max(E-S,0)$ *but* $\quad P(S \le B, t) = 0$

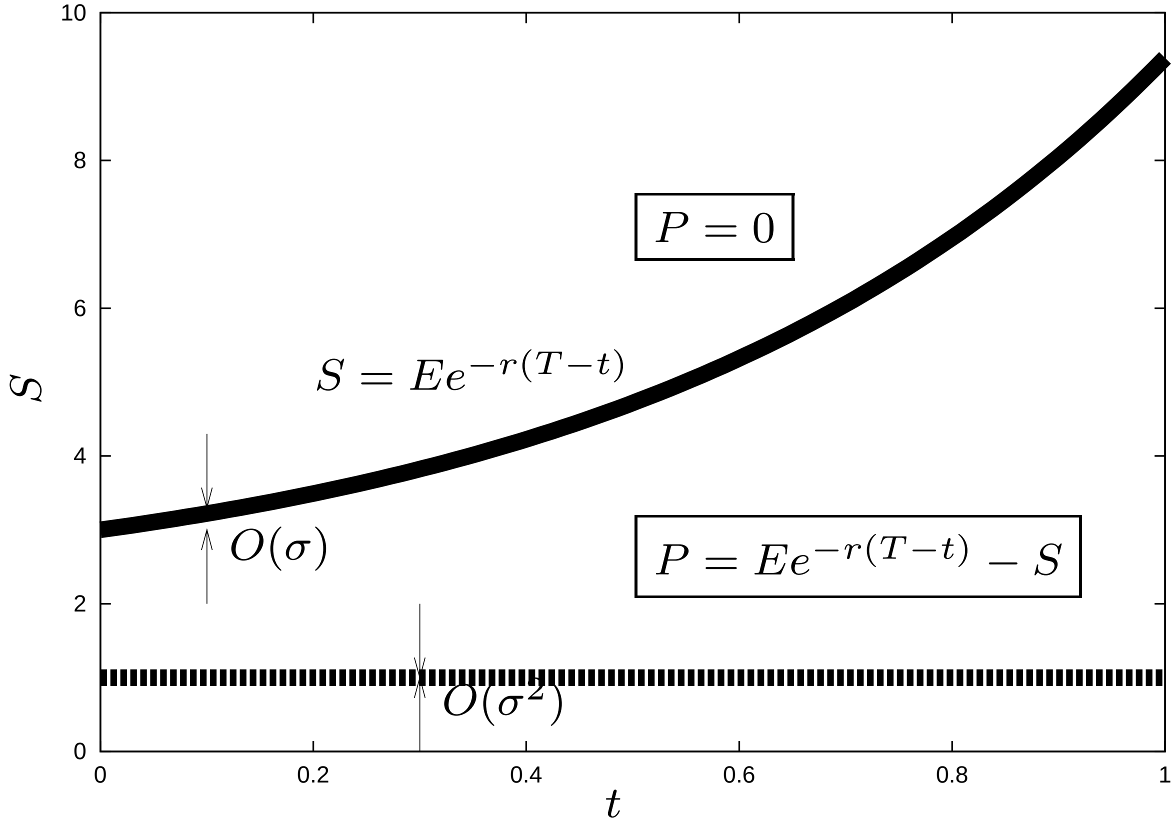
- First assume $Ee^{-rT} > B$ (shear layer does not hit barrier)
- $P \equiv 0$ if $S > Ee^{-r(T-t)}$, $P = Ee^{-r(T-t)} - S$ if $B < S < Ee^{-r(T-t)}$
- Close to $S = Ee^{-r(T-t)}$, $P = \hat{P}/\sigma$, $\hat{S} = (S - E^{-r(T-t)})/\sigma$
- $P(\hat{S}, t)$ similar to European option already described (*shear layer*)
- *Boundary layer* close to $S = B$:
  $S^* = (S-B)/\sigma^2$ (VERY THIN); $P = O(1)$ (to match with above),
  $\sigma^2 S^2 \frac{\partial^2}{\partial S^2} \sim S \frac{\partial}{\partial S}$:

$$\frac{1}{2} B^2 \frac{\partial^2 P}{\partial S^{*2}} + rB \frac{\partial P}{\partial S^*} = 0$$

  Subject to $P(S^* = 0, t) = 0$, $P \to Ee^{-r(T-t)} - B$ as $S^* \to \infty$
- Quasisteady, solution

$$P(S^*, t) = \left[Ee^{-r(T-t)} - B\right] (1 - e^{-\frac{2r}{B} S^*})$$

- No interaction between boundary and shear layers
- Breakdown when $\tau = \frac{T-t}{\sigma^2} = O(1)$ - can be ‘fi xed up’ - additional (τ) derivative,
  $P \to E - B$ as $S^* \to \infty$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$P = 0$
$S = Ee^{-r(T-t)}$
$O(\sigma)$
$P = Ee^{-r(T-t)} - S$
$O(\sigma^2)$

Barrier/down and out (put) options - CASE II

- 'Collision' of shear and boundary layers at t_c, when $Ee^{-r(T-t_c)} = B$
- Define $\tau_1 = \frac{t-t_c}{\sigma^2}$, $S^* = \frac{S-B}{\sigma^2}$ (thin, as case I), $P^* = P/\sigma^2$
- leads to

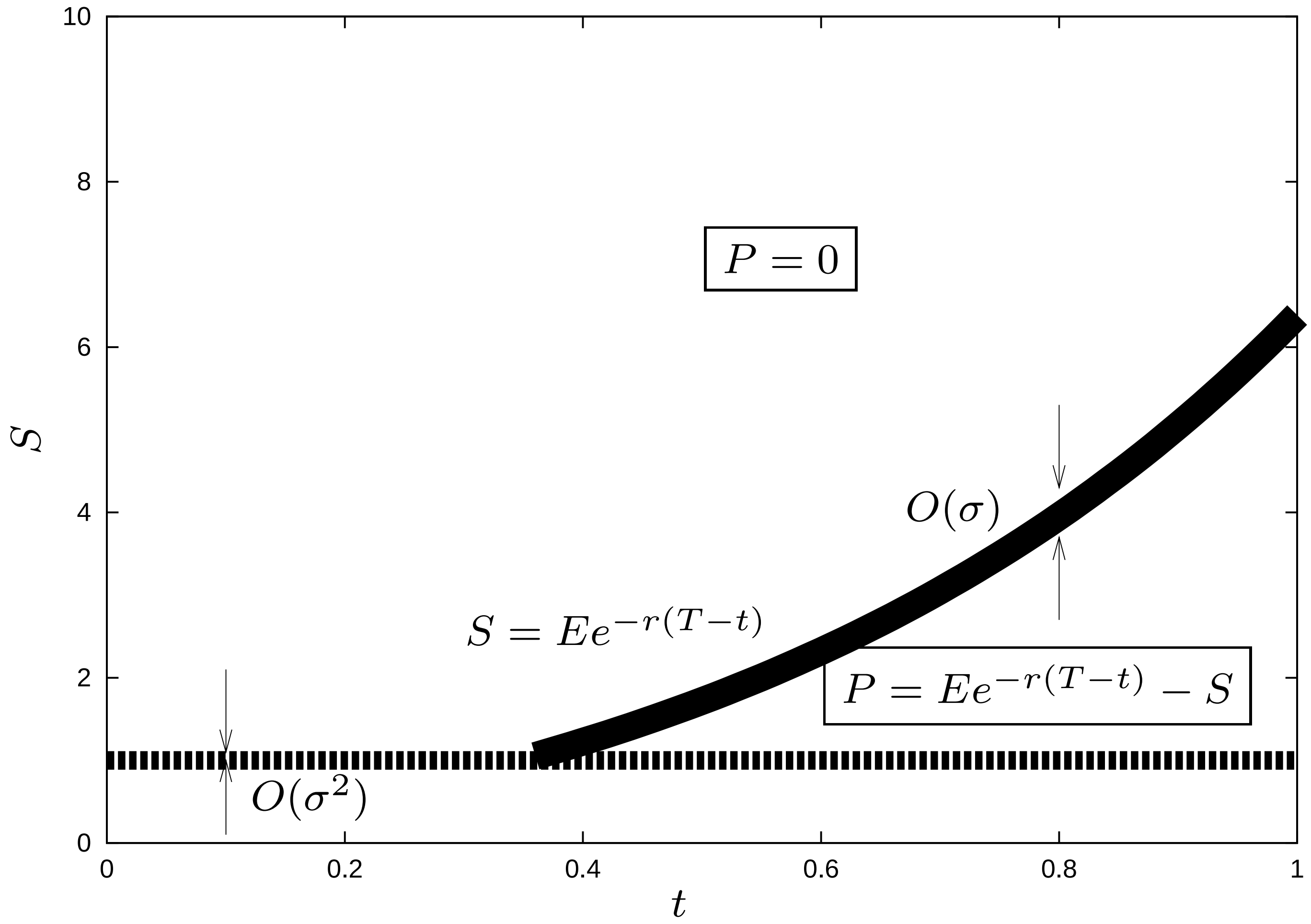
$$\frac{\partial P^*}{\partial \tau_1} + \frac{B^2}{2}\frac{\partial^2 P^*}{\partial S^{*2}} + rB\frac{\partial P^*}{\partial S^*} = 0$$

- Subject to $P^* \to 0$ as $S^* \to \infty$, and, as $\tau_1 \to \infty$:

$$P^* \to Br\tau_1(1 - e^{-\frac{2r}{B}S^*}) \quad S^* = O(1)$$

$$P^* \to \sqrt{\tau_1}B\left[-\eta N(-\eta) + N'(-\eta)\right], \quad \eta = \frac{S^* - Br\tau_1}{B\sqrt{\tau_1}} = O(1)$$

$$P^* = Br\tau_1 - S^*, \quad 0 \ll S^* \ll Br\tau_1$$



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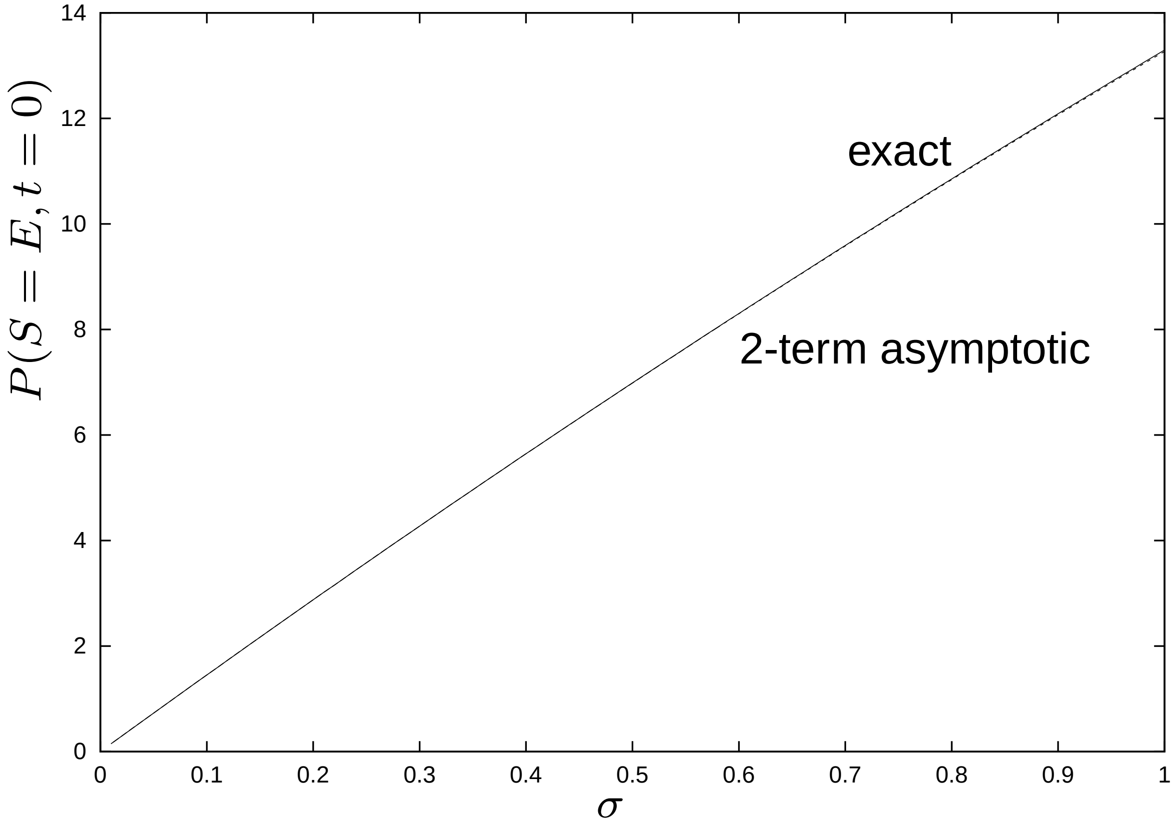
t
$P = 0$
$O(\sigma)$
$S = Ee^{-r(T-t)}$
$P = Ee^{-r(T-t)} - S$
$O(\sigma^2)$

American Options, limit of small volatility ($\sigma \to 0$)

Proceed as before - consider American put

- Set $r = \sigma R$, - small interest rates, comparable to volatility $\implies$ exercise boundary lies inside shear layer
- $P = 0$, $S > E$, $S - E = O(1)$ - worthless at asset prices $O(1)$ above exercise price (c.f. Europeans)
- $P = E - S$, $S < E$, $S - E = O(1)$ - exercise below strike price
- Discontinuity along $S = E$, c.f. Europeans - key regime $\hat{S} = \frac{S-E}{\sigma} = O(1)$
- Exercise along $S = S_f = E + \sigma \hat{S}_{f0}(t) + \sigma^2 \hat{S}_{f1}(t) + \ldots$
  - $P = \sigma \hat{P}_0(\hat{S}, t) + \sigma^2 \hat{P}_1(\hat{S}, t) + \ldots$
  - $O(\sigma^{-1})$: $\mathcal{L}\{\hat{P}_0\} \equiv \frac{1}{2}E^2 \frac{\partial^2 \hat{P}_0}{\partial \hat{S}^2} + RE \frac{\partial \hat{P}_0}{\partial \hat{S}} + \frac{\partial \hat{P}_0}{\partial t} = 0$
  - $\hat{P}_0(\hat{S}_{f0}, t) = -S_{f0}$, $\frac{\partial \hat{P}_0}{\partial \hat{S}}(\hat{S}_{f0}, t) = -1$
  - $O(\sigma^0)$: $\mathcal{L}\{\hat{P}_1\} = -E\hat{S} \frac{\partial^2 \hat{P}_0}{\partial \hat{S}^2} - R\hat{S} \frac{\partial \hat{P}_0}{\partial \hat{S}} + R\hat{P}_0$
  - $\hat{P}_1(\hat{S}_{f0}, t) = 0$, $\frac{\partial \hat{P}_1}{\partial \hat{S}}(\hat{S}_{f0}, t) = \frac{2R}{E} \hat{S}_{f1}$
  - etc.
- Can scale E and R out of problem $(\hat{S}_{f0}, S^*) \to \frac{E}{R}(\hat{S}_{f0}, S^*)$, $t \to \frac{t}{R^2}$ and if $0 \le t \le T \to \infty$ then universal set of results obtained (Widdicks et al, 2005)

American Options - comparison



Comparison of 'exact' and asymptotic results, $S = E = 100$, $r = \sigma$, $T = 0.5$ (body-fi tted coordinates used in both cases)

Multiple underlyings

In the case of an option involving two underlyings

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 V}{\partial S_1^2}+rS_1\frac{\partial V}{\partial S_1}+\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 V}{\partial S_2^2}+rS_2\frac{\partial V}{\partial S_2}+\sigma_1\sigma_2\rho S_1 S_2\frac{\partial^2 V}{\partial S_1 S_2}-rV=0$$

Setting $\sigma_1=\sigma_2=0$ leads to the hyperbolic PDE

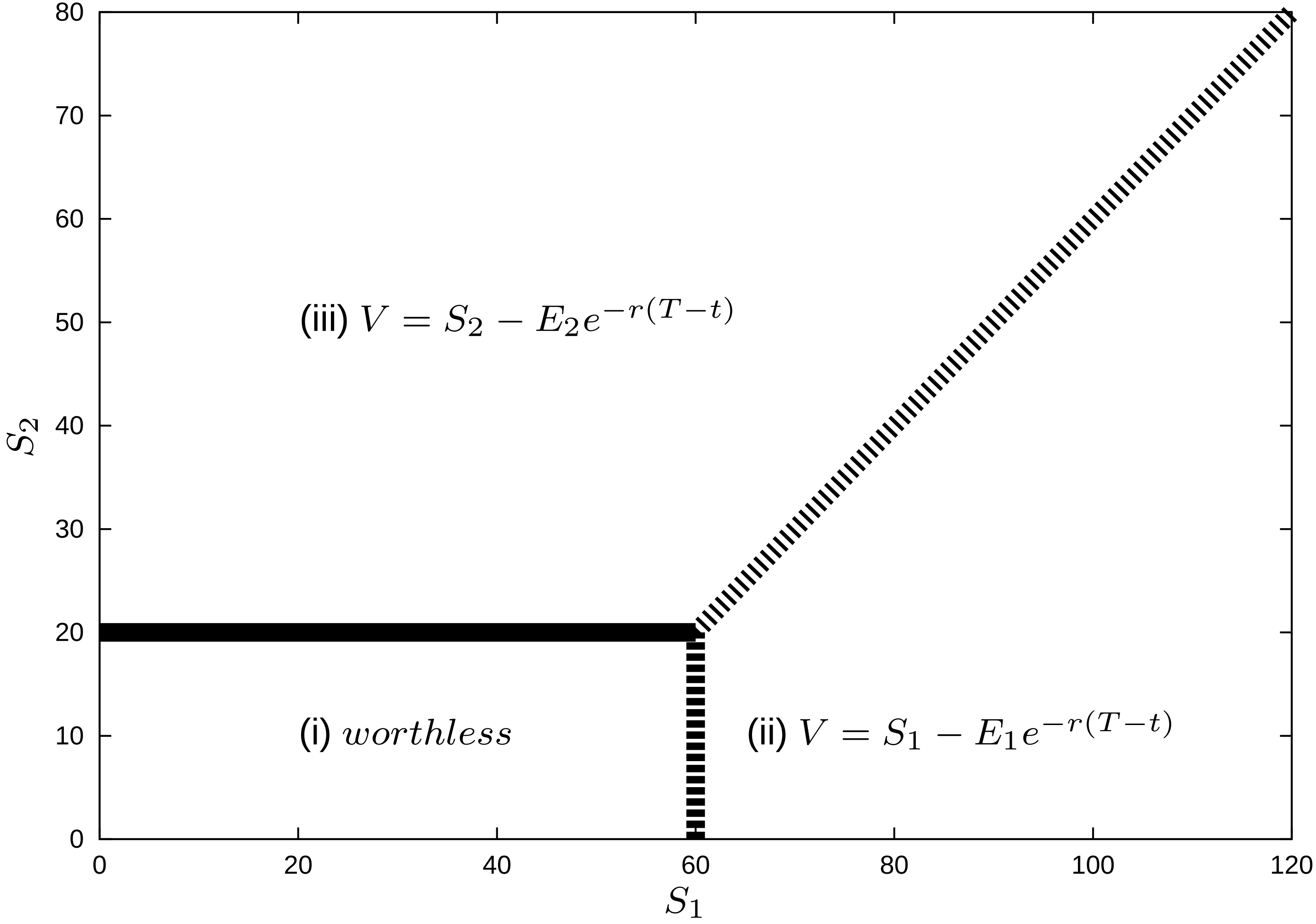
$$\frac{\partial V}{\partial t}+rS_1\frac{\partial V}{\partial S_1}+rS_2\frac{\partial V}{\partial S_2}-rV=0.$$

Many payoff scenarios exist. If

$$V(S_1,S_2,t=T)=\max(S_1-E_1,S_2-E_2,0)\quad\text{(calls)}$$

$$V(S_1,S_2,t=T)=\max(E_1-S_1,E_2-S_2,0)\quad\text{(puts)}.$$

Calls



80
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40
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0
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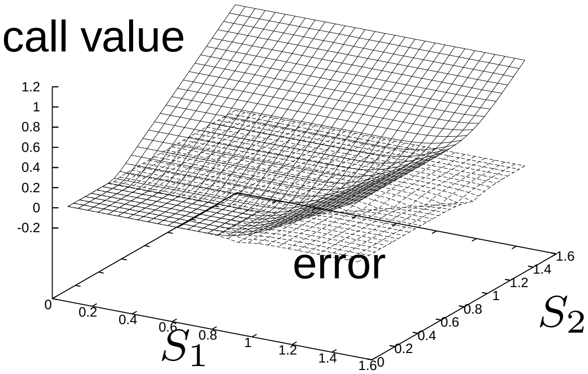
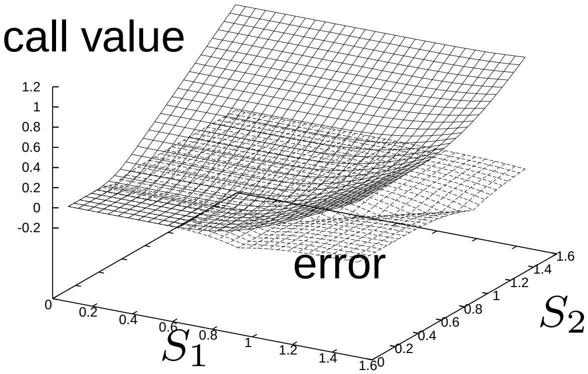
40
60
80
100
120
S_2
S_1
(iii) $V = S_2 - E_2 e^{-r(T-t)}$
(i) $worthless$
(ii) $V = S_1 - E_1 e^{-r(T-t)}$

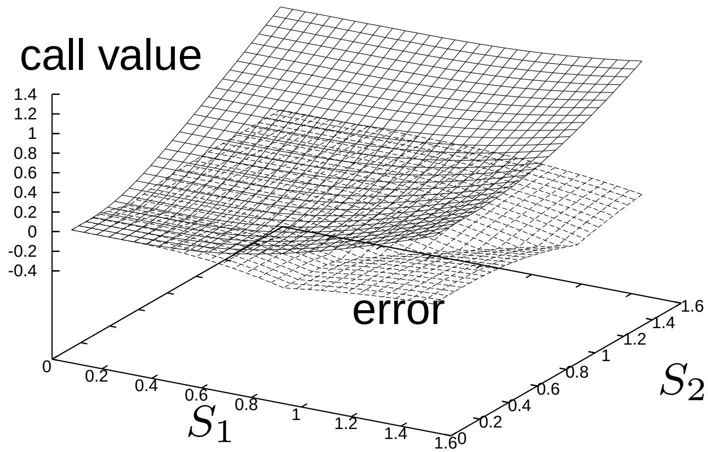
$E_1 = 1$, $E_2 = 0.5$, $r = 0.1$, $\rho = 0.4$, $T = 1$



(a) $\sigma = 0.1$

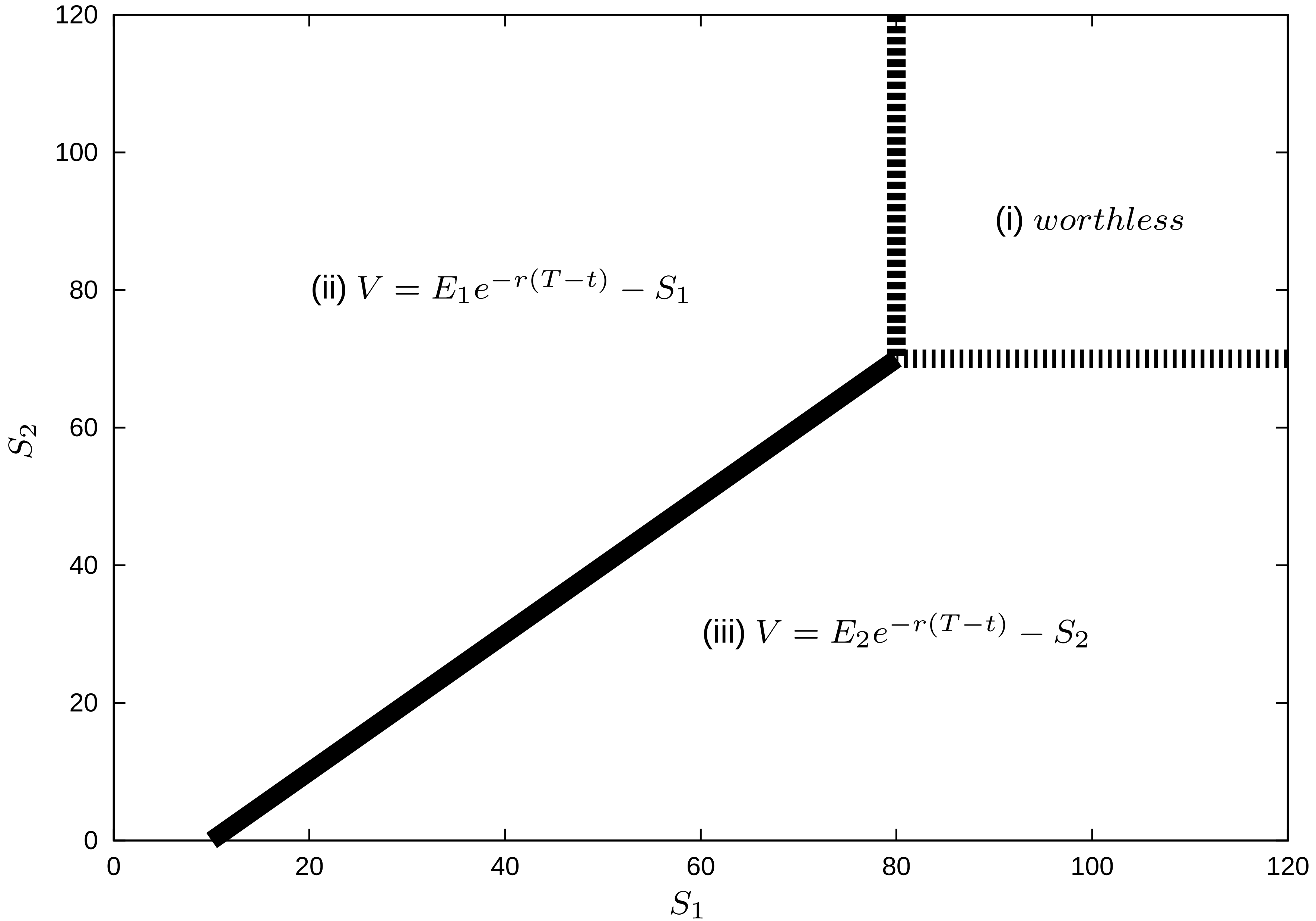


(b) $\sigma = 0.25$



(c) $\sigma = 0.5$

Puts ($E_1 > E_2$)

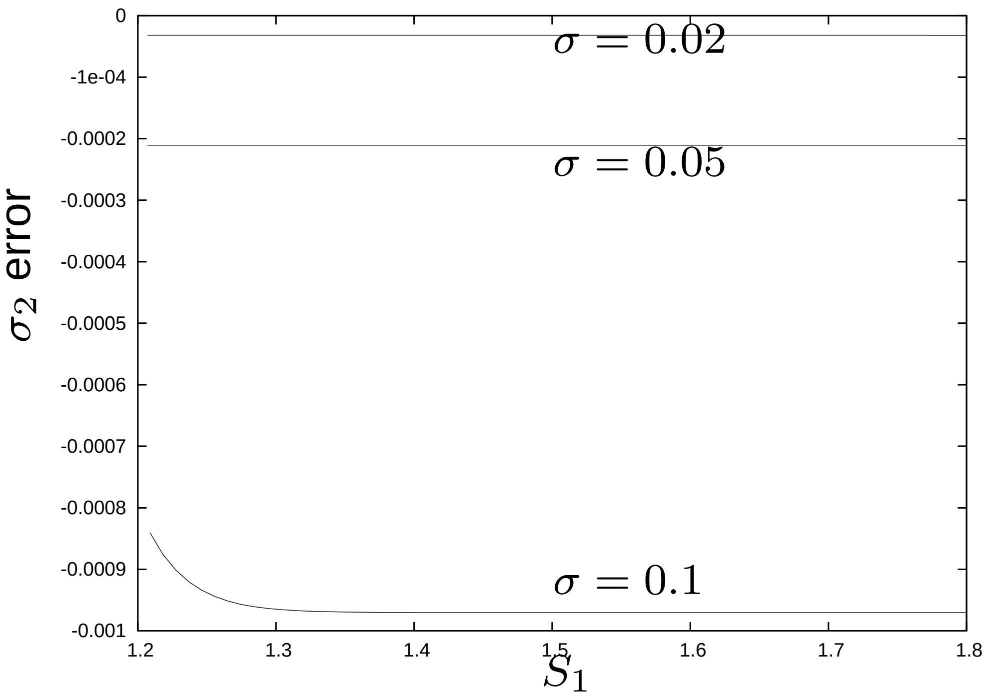


If $E_1 < E_2$ the 45^o line would intersect with the S_2-axis

- Shear layers along $S_1 = E_1 e^{-r(T-t)}$ and $S_2 = E_2 e^{-r(T-t)}$ similar to 1-D case generally
- Somewhat different shear layer along $S_1 - E_1 e^{-r(T-t)} = S_2 - E_2 e^{-r(T-t)}$ but can be reduced to a 1D calculation
- Transition point at $S_1 = E_1 e^{-r(T-t)}$, $S_2 = E_2 e^{-r(T-t)}$.
- Can be extended to incorporate early exercise

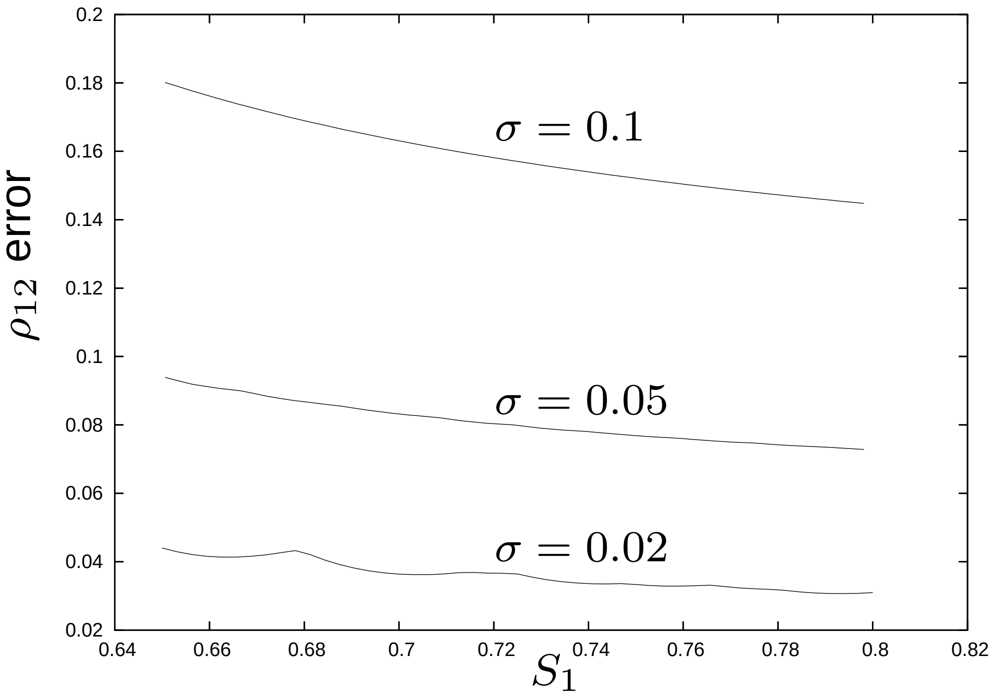
Calculation of implied volatilities

Can use this analysis to more effi ciently back out volatilities and correlation coeffi cients - by choosing 'best' regions of parameter space. Using asymptotic form along $S_2 = E_2$ and comparing with 'exact' calculation (American put, $E_1 = 1$, $E_2 = 0.5$, $T = 1$; in each case $r = \sigma_1 = \sigma_2 = \sigma$, $\rho = 0.4$)



Calculation of implied correlation coefficients

Error in calculation of ρ_{12}, results along $E_1 - S_1 = E_2 - S_2$



Pricing bonds

A canonical equation form for pricing bonds (using several popular interest-rate models) leads to PDEs of the form (arising from stochastic modelling of interest rates)

$$\frac{1}{2}\sigma^2(r,t)r^{2\beta}\frac{\partial^2 F}{\partial r^2} + \left[f(r,t) + \sigma(r,t)r^{\beta}\lambda(r,t)\right]\frac{\partial F}{\partial r} - rF + \frac{\partial F}{\partial t} = 0$$

For a bond, the final condition is $F(r, t = T) = F_f(r)$.
For example, for the CIR model
$f(r,t) = \kappa(\theta - r), \quad \beta = \frac{1}{2}, \quad \lambda = -\hat{\lambda}\sqrt{r}/\sigma.$
This has a (messy) exact analytic solution. Let us use small perturbation theory instead. Setting $\sigma = 0$ in the full equation, leads to (first-order) equations of the generic form ($\tau = T - t$):

$$(A_0 - B_0 r)\frac{\partial F_0}{\partial r} - rF_0 - \frac{\partial F_0}{\partial \tau} = 0$$

Solution of reduced problem

Equation can be solved very easily using the method of characteristics, i.e.

$$F_0(r,\tau) = F_f\left(\frac{A_0}{B_0} - (\frac{A_0}{B_0} - r)e^{-B_0\tau}\right)\exp\left(\frac{1}{B_0^2}(A_0 - B_0 r)(1 - e^{-B_0\tau}) - \frac{A_0}{B_0}\tau\right)$$

In the case of options on these bonds, (horrible) exact solutions do exist, but the small perturbation method is much simpler - and surprisingly accurate even for quite big σ's. For a call (maturing at $t = T_0$, $\tau = T - T_0$) on a bond (maturing at $t = T$, $\tau = T$)

$$\begin{aligned} C_0(r,\tau) &= \max\left(F_f \exp\left(\frac{1}{B_0^2}(A_0 - B_0 r_1)(1 - e^{-B_0(T-T_0)}) - \frac{A_0}{B_0}(T - T_0)\right) - X, 0\right) \\ &\times \exp\left(\frac{1}{B_0^2}(A_0 - B_0 r)(1 - e^{-B_0\tau_1}) - \frac{A_0}{B_0}\tau_1\right), \end{aligned}$$

where

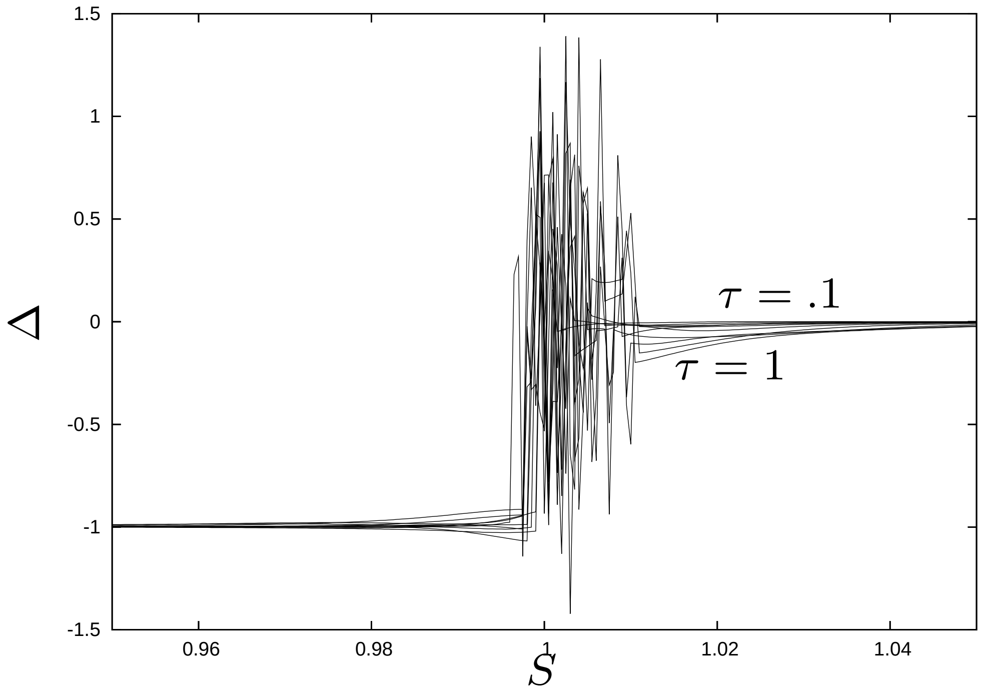
$$r_1 = \frac{A_0}{B_0} - (\frac{A_0}{B_0} - r)e^{-B_0\tau_1}, \tau_1 = \tau - (T - T_0)$$

Illiquid markets

Modifi ed Black-Scholes model (c.f. Liu & Yong, 2005, without fudge factor), with feedback - a seriously nonlinear PDE - an example where insight gleaned from asymptotics is invaluable.

$$-\frac{\partial P}{\partial \tau} + \frac{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}}{(1 - \rho \frac{\partial^2 P}{\partial S^2})^2} + rS\frac{\partial P}{\partial S} - rP = 0$$

Consider the European put version of this option (using my favourite Crank-Nicolson method)

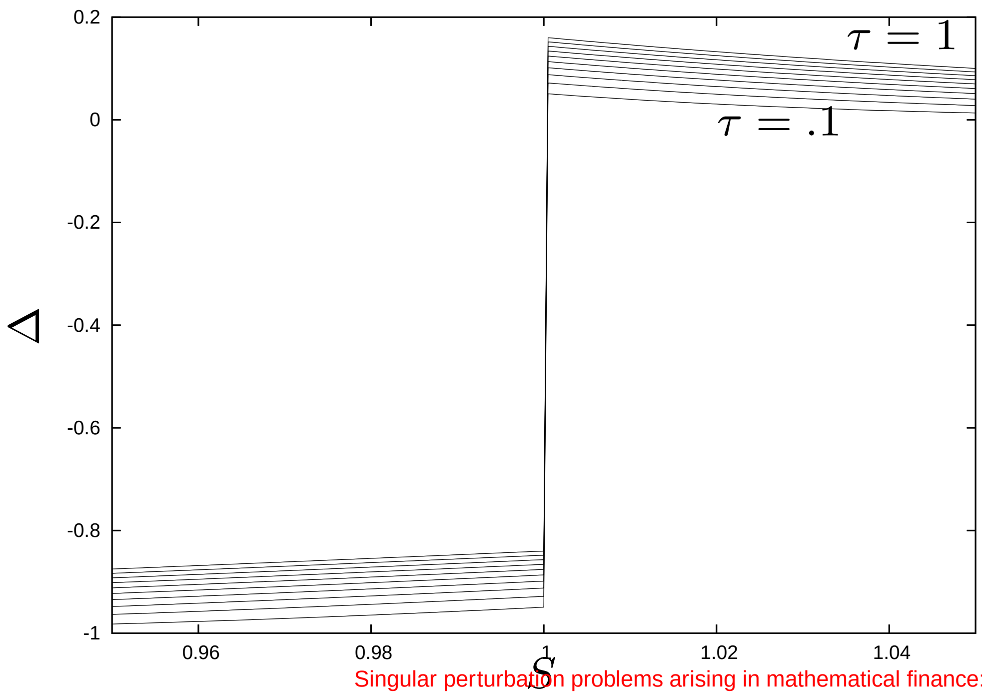


Asymptotics close to expiry ($\tau \to 0$)

Solution takes the form $P \to -\tau^{\frac{1}{2}}\hat{S}H(-\hat{S}) + \tau\hat{P}_0(\hat{S}) + \ldots$, where $\hat{S} = (S-E)/\tau^{\frac{1}{2}}$ is the key (shear-layer) scaling, close to strike price ($H(-\hat{S})$ the Heaviside function)

$$-\hat{P}_0 + \frac{1}{2}\hat{S}\frac{\partial \hat{P}_0}{\partial \hat{S}} + \frac{\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \hat{P}_0}{\partial \hat{S}^2}}{(1-\rho\frac{\partial^2 \hat{P}_0}{\partial \hat{S}^2})^2} - rH(-\hat{S}) = 0$$

where $\hat{P}_0$ and $\frac{\partial \hat{P}_0}{\partial \hat{S}}$ continuous at $\hat{S}=0$, and $\hat{P}_0 + H(-\hat{S})rE \to 0$ as $|\hat{S}| \to \infty$.
Discontinuity in the Δ at $S = E$ of +1 for all time - can build jump condition into full calculation (using Keller-Cebeci box scheme)



Conclusions

- Singular perturbation techniques can provide quick, simple solutions across wide regions of parameter space.
- Along (thin) zones, discontinuities may occur, but can be resolved by blending (i.e. boundary or shear) layers.
- Gives insight into fi nancial product pricing
- Potentially quite universal tool for rapid (and surprisingly accurate) pricing of fi nancial products described by Black-Scholes-like PDEs.